STEADY-STATE VIBRATIONS OF PLATES WITH FREE EDGES

(USTANOVIVSHIESIA KOLEBANIIA PLASTINOK SO SVOBODNYMI KRAIAMI)

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A.I.LIKHODED

(Moscow)

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This investigation is concerned with the forced, periodic vibration of a homogeneous, isotropic, uniformly thick plate with free edges. It is assumed that the plate consists of an arbitrary shaped, simply connected region bounded by a curve with differentiable curvature. On the basis of considerations given in [1 and 2], the problem is reduced to a Fredholm integral equation of the second kind. The kernel of the resultant integral equation is expressed in terms of known special functions. The existence of a solution is investigated.

1. Assume that the plate occupies an arbitrary, simply connected region S in the plane $z = x + iy$, and that the curvature of the bounding curve L is everywhere differentiable. The coordinate origin is taken to be inside the region S. The amplitude of the forced vibrations will be written as a sum $w(x, y) = u(x, y) + u_0(x, y)$, where $u_0(x, y)$ is the particular solution of the equation, and $u(x,y)$ must satisfy Equation

$$\triangle\triangle u - \lambda^4 u = 0 \qquad \left(\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.1}$$

with the following boundary conditions (*):

$$M(u) \equiv \sigma\triangle u + (1-\sigma)\left[\cos^2\theta\,\frac{\partial^2 u}{\partial x^2} + \sin^2\theta\,\frac{\partial^2 u}{\partial y^2} + \sin 2\theta\,\frac{\partial^2 u}{\partial x\partial y}\right] = -M(u_0)$$

$$Q(u) \equiv \frac{d\triangle u}{dn} + (1-\sigma)\frac{d}{ds}\left[\cos 2\theta\,\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{2}\sin 2\theta\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)\right] = -Q(u_0) \tag{1.2}$$

Here, λ is related to the frequency, θ is the angle between the outward normal and the x-axis and σ is Poisson's ratio. The first condition in (1.2) implies the vanishing of the bending moment on the boundary L; the second condition implies the vanishing of the transverse shear resultant. We will seek a solution $u(x,y)$ in integral form

$$u(x, y) = \int_L [\nu_1(s) F_1(s, x, y) + \nu_2(s) F_2(s, x, y)]\, ds \tag{1.3}$$

Here $\nu_1(s)$ and $\nu_2(s)$ are unknown density functions; the kernel functions $F_j(s,x,y)$ may be expressed as

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*) For the static deflection case, the boundary conditions become simplified [3 and 4].

$$F_j(s, x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} [g_{j,1}(\alpha) e^{\gamma_1(\alpha)} + g_{j,2}(\alpha) e^{\gamma_2(\alpha)}] \, d\alpha \quad (j = 1, 2) \tag{1.4}$$

$$\gamma_1(\alpha) = i\alpha c_1 - c_2 \sqrt{\alpha^2 + \lambda^2}, \quad \gamma_2(\alpha) = i\alpha c_1 - c_2 \sqrt{\alpha^2 - \lambda^2}, \quad \gamma^* = i\alpha t' \overline{(t - z)}$$

$$c_1 = \mathrm{Re}\, [t' \overline{(t - z)}], \qquad c_2 = \mathrm{Im}\, [t' \overline{(t - z)}] \quad (t = \xi + i\eta)$$

where t is a generic boundary point and the prime denotes differentiation with respect to s . The radicals in the preceding equations are either positive real or positive imaginary. It is not difficult to show that the function $u(x,y)$ in (1.3) satisfies Equation (1.1). The coefficients $g_{i,k}(\alpha)$ are later chosen in such a manner that substitution of $u(x,y)$ into the boundary conditions leads to Fredholm integral equations of the second kind.

The boundary conditions will now be transformed so as to simplify the procedure of obtaining the coefficients $g_{i,k}(\alpha)$, By differentiating the first condition in (1.2) with respect to s and combining the result with the expression for the moment $M[w(t_a)]$ at a certain point t_a on the boundary L , we obtain (when the orders of the highest-order derivatives are the same in both boundary conditions the procedure of obtaining the coefficients $g_{i,k}(\alpha)$ becomes simplified [2])

$$\frac{dM\,[w]}{ds} + M\,[w\,(t_a)] = 0 \tag{1.5}$$

Since the solution belongs to the class of single-valued functions, integration of (1.5) with respect to s , taken along the boundary L , establishes the equivalence of (1.5) with the first condition in (1.2). By using (1.5) together with the second condition in (1.2), the boundary conditions are formulated in terms of third-order differential operators.

N o t e . The following two integral relations for the vibration amplitude may be obtained from (1.2)

$$\int_L Q\,(w)\, ds = \int_L \frac{d\triangle w}{dn}\, ds = 0, \qquad \int_L M\,(w)\, ds = 0 \tag{1.6}$$

Transforming the contour integrals in (1.6) into area integrals and noting that $w(x,y)$ must satisfy Equation $\triangle\triangle w - \lambda^4 w = p\,(x, y)$, we obtain

$$\lambda^4 \iint_S w\,(x, y)\, dS = -\iint_S p\,(x, y)\, dS, \qquad \lambda^4 \iint_S zw\,(x, y)\, dS = -\iint_S zp\,(x, y)\, dS \tag{1.7}$$

Here $p(x,y)$ is the amplitude of the distributed load divided by the flexural rigidity of the plate. The first relation in (1.7) pertains to the displacement amplitude of the mass center of the plate; the second pertains to the angular rotation of the plate as a rigid body. For $\lambda = 0$, (1.7) yields the necessary conditions for the existence of a solution in the case of static deflection of the plate [4]. We will assume that the resultant force and moment (right-hand sides in (1.7)) are equal to zero (the general case is reducible to the one under discussion).

We will write the functions $F_1(s,x,y)$ and $F_2(s,x,y)$ in the form

$$F_j(s, x, y) = \frac{2}{\pi}\, \mathrm{Re} \int_0^D [g_{j,1} e^{\gamma_1(\alpha)} + g_{j,2} e^{\gamma_2(\alpha)}]\, d\alpha + \tag{1.8}$$

$$+ \frac{2}{\pi}\, \mathrm{Re} \int_D^\infty [g_{j,1} e^{\gamma_1(\alpha)} + g_{j,2} e^{\gamma_2(\alpha)}]\, d\alpha \quad (j = 1, 2)$$

Here D is a sufficiently large positive number. The singularities of the functions $F_j\,(s,x,y)$ are contained in the second terms. For large values of α , the coefficients $g_{i,k}$ may be expanded

$$g_{11}=i\frac{k_1}{\alpha}+i\frac{k_2}{\alpha^3}+\cdots,\quad g_{1,2}=-i\frac{k_1}{\alpha}+i\frac{k_2}{\alpha^3}+\cdots,\quad g_{21}=-\frac{k_1}{\alpha}-\frac{(1+\sigma)k_2}{2\alpha^3}+\cdots$$
$$g_{22}=\frac{k_1}{\alpha}-\frac{(1+\sigma)k_2}{2\alpha^3}+\cdots,\quad k_1=\frac{1}{2\lambda^2(1-\sigma)},\quad k_2=\frac{1}{2(1-\sigma)^2} \tag{1.9}$$

In order to investigate the singularities of the functions $F_1(s,x,y)$ and $F_2(s,x,y)$, use will be made of the relations

$$\int_D^\infty \frac{e^{\gamma^*(\alpha)}d\alpha}{\alpha^3}=\frac{1}{2}t'^2\overline{(z-t)^2}\,\overline{\ln(t-z)}+\cdots,\quad \int_{-\infty}^{\infty}\frac{e^{\gamma_2(\alpha)}d\alpha}{\sqrt{\alpha^2-\lambda^2}}=-i\pi H_0^{(2)}(\lambda r) \tag{1.10}$$

$$\operatorname{Im}\int_D^\infty \frac{e^{\gamma_1(\alpha)}}{\alpha}d\alpha=I_0(\lambda r)\Phi+\cdots,\quad \operatorname{Im}\int_D^\infty\frac{e^{\gamma_2(\alpha)}}{\alpha}d\alpha=J_0(\lambda r)\Phi+\cdots,\quad \Phi=\tan^{-1}\frac{\eta-y}{\xi-x}$$
$$(r=\sqrt{(x-\xi)^2+(y-\eta)^2})$$

Here $H_0^{(2)}(\lambda r)$ is the Hankel function of the second kind while $J_0(\lambda r)$ and $I_0(\lambda r)$ are the ordinary and modified Bessel functions of the first kind, respectively. The dots represent terms having continuous third derivatives.

From (1.9) and (1.10), we obtain the principal parts of the functions $F_1(s,x,y)$ and $F_2(s,x,y)$; denoting these by Γ_1 and Γ_2, we have

$$\Gamma_1=-\frac{2k_2}{\pi}\left\{\operatorname{Im}\left[t'^2\overline{(z-t)^2}\,\overline{\ln(t-z)}\right]+\frac{1-\sigma}{2}r^2\Phi\right\}$$
$$\Gamma_2=-\frac{k_2}{\pi}\left\{(1+\sigma)\operatorname{Re}\left[t'^2_s\overline{(z-t)^2}\,\overline{\ln(t-z)}\right]+(1-\sigma)r^2\ln r\right\} \tag{1.11}$$

2. Introduce the complex density function $\omega(t)=\nu_1+i\nu_2$, then the principal part of the function $u(x,y)$ in (1.3) may be written

$$u^*(x,y)=\frac{1}{2}\int_L\left[\omega(t)(\Gamma_1-i\Gamma_2)+\overline{\omega(t)}(\Gamma_1+i\Gamma_2)\right]ds \tag{2.1}$$

We now combine the boundary conditions (1.2) into the following single complex equation

$$G(u)\equiv M(u)+i\int^s Q(u)\,ds\equiv 2(1-\sigma)\left[\varkappa\frac{\partial^2 u}{\partial t\partial\bar t}-\bar t'^2\frac{\partial^2 u}{\partial\bar t^2}\right]-8\int^{\bar t}\frac{\partial^3 u}{\partial t\partial\bar t^2}\,d\bar t=f(t)$$
$$f(t)=-M(u_0)-i\int^s Q(u_0)\,ds,\qquad \varkappa=\frac{3+\sigma}{1-\sigma} \tag{2.2}$$

Taking into account the fact that relation (2.2) is obtainable from condition (1.5) and the second condition in (1.2) by means of integration with respect to s along the boundary, we arrive at the following new density function in (2.1):

$$\Omega(t)=\int_{t_a}^{t}\omega(t)\,dt$$

Integrating by parts and neglecting the terms which contain no singularities, we arrive at the following modified representation of the principal part of $u(x,y)$

$$u^*(x,y)=-\frac{1}{2\pi}\operatorname{Im}\left\{\int_L\Omega(t)\,\overline{(t-z)}\left[\ln\left(1-\frac{z}{t}\right)+\varkappa\overline{\ln\left(1-\frac{z}{t}\right)}-\varkappa\right]dt\right\} \tag{2.3}$$

We will assume that $\Omega(t)$ satisfies a Hölder condition on L. The analytic functions $\varphi(z)$ and $\chi(z)$ which, in accordance with Goursat's formula, correspond to the biharmonic function $u^*(x,y)$ are equal to

$$\varphi(z)=\frac{1}{2\pi i}\int_L \Omega(t)\ln\left(1-\frac{z}{t}\right)dt, \qquad \psi(z)=\chi'(z)$$

$$\chi(z)=\frac{1}{2\pi i}\left\{\varkappa\int_L \overline{\Omega(t)}(\bar t-z)\left[\ln\left(1-\frac{z}{t}\right)-1\right]\overline{dt}-\int_L \Omega(t)\,\bar t\,\ln\left(1-\frac{z}{t}\right)dt\right\} \tag{2.4}$$

Introduce the functions

$$\Pi_1(\lambda,t,z)=-\frac{2}{\lambda^2}\left[N(\lambda r)+K(\lambda r)+(J_0(\lambda r)-I_0(\lambda r))(i\Phi-\ln t)\right]$$

$$\Pi_2(\lambda,t,z)=\frac{1}{2}\left[N(\lambda r)-K(\lambda r)+(J_0(\lambda r)+I_0(\lambda r))(i\Phi-\ln t)\right] \tag{2.5}$$

$$N(\lambda r)=N_0(\lambda r)-J_0(\lambda r)(\ln {}^1\!/_2\lambda+C),\quad K(\lambda r)=K_0(\lambda r)+I_0(\lambda r)(\ln {}^1\!/_2\lambda+C)$$

Here $N_0(\lambda r)$ is the Bessel function of second kind of order zero, $K_0(\lambda r)$ is the modified Bessel function of second kind, and C is Euler's constant. The functions $\Pi_i(\lambda,t,z)$ $(i=1,2)$ satisfy Equation (1.1), and may be written in the form

$$\Pi_1(\lambda,t,z)=r^2\left[\ln\left(1-\frac{z}{t}\right)-1\right]+P_1(\lambda,t,z)$$

$$\Pi_2(\lambda,t,z)=\ln\left(1-\frac{z}{t}\right)+P_2(\lambda,t,z) \tag{2.6}$$

Here $P_1(\lambda,t,z)$ and $P_2(\lambda,t,z)$ are entire functions of the parameter λ which vanish when $\lambda=0$ and which have third order continuous derivatives with respect to t and z. In addition, introduce the two functions

$$E_1(\lambda,t,t_0)=P_1(\lambda,t,t_0)+\varkappa\overline{P_1(\lambda,t,t_0)}-\frac{4}{\lambda^2}\left[N(\lambda\rho_0)+K(\lambda\rho_0)\right]$$

$$E_2(\lambda,t,t_0)=P_2(\lambda,t,t_0)+\varkappa\overline{P_2(\lambda,t,t_0)}+N(\lambda\rho_0)-K(\lambda\rho_0),\quad \rho_0^2=(t_0-\beta)(\overline{t_0}-\overline{\beta}) \tag{2.7}$$

Here β is some fixed point in the z-plane, not lying on L. By taking into account (2.6), the sought function $u(x,y)$ whose principal part is $u^*(x,y)$ is easily constructed. This function takes the form

$$u(x,y)=\frac{1}{2\pi}\operatorname{Im}\left\{\int_L \Omega(t)\frac{\partial}{\partial z}\left[\Pi_1(\lambda,t,z)+\varkappa\overline{\Pi_1(\lambda,t,z)}\right]dt-A(\Omega)\frac{\partial V(\lambda\rho)}{\partial z}\right\}-$$

$$-\frac{V(\lambda r_0)}{2\pi i B(\lambda)}\int_L\left[\Omega(t)h_1(\lambda,t)-\overline{\Omega(t)}h_2(\lambda,t)\right]dt+\frac{8V(\lambda r_0)}{(1-\sigma)B(\lambda)}\int_L\frac{\partial^3 u_0}{\partial t\partial\bar t^2}\,\bar t'^2\,dt \tag{2.8}$$

where the following notation has been used

$$A(\Omega)=\int_L\Omega(t)\,dt,\quad B(\lambda)=\frac{8}{1-\sigma}\int_L\frac{\partial^3 V(\lambda r_0)}{\partial t\partial\bar t^2}\,\bar t'^2dt \qquad \left(B(0)=\frac{16\pi i}{1-\sigma}\right)$$

$$V(\lambda r_0)=\frac{4}{\lambda^2}\left[N(\lambda r_0)+K(\lambda r_0)\right],\qquad r_0^2=z\bar z,\qquad \rho^2=(z-\beta)(\overline{z-\beta})$$

$$h_1(\lambda,t)=\frac{\lambda^4}{4(1-\sigma)}\int_L E_1(\lambda,t,t_0)\,\bar t_0'^2dt_0,\qquad h_2(\lambda,t)=\frac{4\bar t'^2}{1-\sigma}\int_L\frac{\partial^2\overline{E_2(\lambda,t,t_0)}}{\partial\bar t_0^2}\bar t_0'^2dt_0$$

The right-hand side of (2.8) holds for all values of λ except for the zeros of $B(\lambda)$.

Utilizing the Sokhotski-Plemelj formulas, we obtain the limiting values of the derivatives $\partial^2u/\partial z\partial\bar z$, $\partial^2u/\partial\bar z^2$ and $\partial^3u/\partial z\partial\bar z^2$ as z approaches the boundary point t_0. Substituting these values into the boundary condition (2.2), we arrive at the following Fredholm integral equation of the second kind

$$\varkappa\Omega(t_0)+\frac{1}{2\pi i}\int_L\left[\Omega(t)G_1(\lambda,t,t_0)+\overline{\Omega(t)}G_2(\lambda,t,t_0)\right]dt=f_1(t_0) \tag{2.9}$$

Here

$$G_1(\lambda,\ t,\ t_0) = \frac{d}{dt}\ln\frac{t-t_0}{\bar t-\bar t_0} + \varkappa\frac{\bar t'^2-\bar t_0'^2}{\bar t-\bar t_0} - \varkappa\frac{\partial E_2(\lambda,\ t,\ t_0)}{\partial t_0} + \bar t_0'^2\frac{\partial E_2(\lambda,\ t,\ t_0)}{\partial \bar t_0} + $$

$$+\frac{\lambda^4}{4(1-\sigma)}\int^{t_0} E_1(\lambda,\ t,\ t_0)\,\bar t_0'^2 dt_0 + \frac{G[V(\lambda r_0)]}{(1-\sigma)B(\lambda)}h_1(\lambda,\ t)$$

$$G_2(\lambda,\ t,\ t_0) = \bar t_0'^2\frac{d}{dt}\,\frac{t-t_0}{\bar t-\bar t_0} + \frac{\bar t'^2-\bar t_0'^2}{\bar t-\bar t_0} + \bar t'^2\left[\varkappa\frac{\partial\overline{E_2(\lambda,\ t,\ t_0)}}{\partial\bar t_0} - \bar t_0'^2\frac{\partial^3\overline{E_1(\lambda,\ t,\ t_0)}}{\partial\bar t_0^3} - \right.$$

$$\left. - \frac{4}{1-\sigma}\int^{t_0}\frac{\partial^2\overline{E_2(\lambda,\ t,\ t_0)}}{\partial\bar t_0^2}\bar t_0'^2 dt_0\right] - \frac{G[V(\lambda r_0)]}{(1-\sigma)B(\lambda)}h_2(\lambda,\ t)$$

$$f_1(t_0) = -\frac{f(t_0)}{1-\sigma} - \frac{G[V(\lambda r_0)]}{(1-\sigma)^2 B(\lambda)}\int_L \frac{\partial^3 u_0}{\partial t\partial\bar t^2}\bar t'^2 dt$$

The kernels and the right-hand side of the above equation are continuous functions of t and t_0 (it easily seen that their continuity is guaranteed by the presence of the last two terms in (2.8)). We will now show that when $\lambda = 0$ Equation (2.9) has a unique solution. To prove this assertion, it is sufficient to show that the corresponding homogeneous equation

$$\varkappa\Omega(t_0) + \frac{1}{2\pi i}\int_L \Omega(t)\,d\ln\frac{t-t_0}{\bar t-\bar t_0} + \frac{\bar t_0'^2}{2\pi i}\int_L\overline{\Omega(t)}\,d\frac{t-t_0}{\bar t-\bar t_0} +$$

$$+\frac{1}{2\pi i}\int_L[\varkappa\Omega(t)+\overline{\Omega(t)}]\,\frac{\bar t'^2-\bar t_0'^2}{\bar t-\bar t_0}dt + R[t_0,\ A(\Omega)] = 0 \tag{2.10}$$

has only the trivial solution. Define $R[t,\ A(\Omega)]$

$$R[t,\ A(\Omega)] = \frac{1-\sigma}{2\pi i}\left\{A(\Omega)\left[\frac{\varkappa}{t_1} - \frac{\bar t'^2}{\bar t_1}\right] + \overline{A(\Omega)}\left[\frac{1}{\bar t_1} - \varkappa\frac{\bar t'^2 t_1}{\bar t_1^2}\right]\right\},\quad t_1 = t - \beta$$

Let $\Omega_0(t)$ be a solution of Equation (2.10); the corresponding functions $\varphi(z)$ and $\psi(z)$ in (2.4) will be denoted by $\varphi_0(z)$ and $\psi_0(z)$. Then Equation (2.10) may be written as:

$$\frac{d}{dt}[\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}] + R[t,\ A(\Omega_0)] = 0 \tag{2.11}$$

Integrating (2.11) with respect to t along the closed contour L and taking into account Equations

$$\int_L\left[\frac{\varkappa}{t_1} - \frac{\bar t'^2}{\bar t_1}\right]dt = \frac{2i\delta(\beta)}{1-\sigma},\qquad \int_L\left[\varkappa\frac{\bar t'^2 t_1}{\bar t_1^2} - \frac{1}{\bar t_1}\right]dt = 2\,\frac{1+\sigma}{1-\sigma}\int_L\frac{dt}{\bar t_1} \tag{2.12}$$

we obtain

$$i\delta(\beta)A(\Omega_0) - (1+\sigma)\overline{A(\Omega_0)}\int_L\frac{dt}{\bar t_1} = 0,\qquad \delta(\beta) = \begin{cases} 4\pi, & \text{if } \beta \text{ is inside } S \\ 0, & \text{if } \beta \text{ is outside } S\end{cases} \tag{2.13}$$

By combining (2.13) with its conjugate, we obtain a system of homogeneous linear equations in $A(\Omega_0)$ and $\overline{A(\Omega_0)}$. In order that the determinant of the system be nonzero, we must have

$$\left|\int_L\frac{\bar t'^2 dt}{t-\beta}\right| \neq \frac{\delta(\beta)}{1+\sigma} \tag{2.14}$$

The integral in (2.14) is a function of β, so that (2.14) can obviously always be satisfied by an appropriate choice of β. Upon satisfying (2.14), we have $A(\Omega_0) = 0$

From the above considerations, it is clear that (2.11) is equivalent to the following two equations:

$$\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} = \text{const}, \qquad \int_L \Omega_0(t)\,dt = 0 \tag{2.15}$$

Integrating by parts the expressions for the functions $\varphi_0(z)$ and $\psi_0(z)$ and taking into account the second relation in (2.15), we obtain

$$\varphi_0(z) = -\frac{1}{2\pi i}\int_L \Omega_1(t)\left(\frac{1}{t-z} - \frac{1}{t}\right)dt, \qquad \Omega_1(t) = \int_{t_a}^{t} \Omega_0(t)\,dt$$

$$\psi_0(z) = \frac{\varkappa}{2\pi i}\int_L \overline{\Omega_1(t)}\left(\frac{1}{t-z} - \frac{1}{t}\right)dt + \frac{1}{2\pi i}\int_L \Omega_0(t)\frac{\bar{t}dt}{t-z} \tag{2.16}$$

Then, proceeding as in [5 and 6], we can show that $\Omega_1(t) = \text{const}$, and, consequently, $\Omega_0(t) = 0$ everywhere in L. Whence, based on Tamarkin's theorem [7], the existence of a solution of the integral equation (2.9) follows for nearly all values of the parameter λ.

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